1 Abstract

In this paper, we calculate the automorphism of the Taxicab group which is a transformation group preserving the Taxicab metric. In the process, we showed that the automorphism of $\mathbb{R}$ with the standard addition is the set of all linear transformations with respect to $\mathbb{R}$ over $\mathbb{Q}$. Furthermore, the $\text{Aut}(\mathbb{R})$ is isomorphic to the ring of $\mathbb{R} \times \mathbb{R}$ column-finite matrices with entries in $\mathbb{Q}$ . We go further to characterize the center of $\text{Aut}(\mathbb{R})$ and show $\text{GL}(n, \mathbb{Q})$ is a subgroup of $\text{Aut}(\mathbb{R})$ for all finite $n$.

2 The Taxicab Group

The Taxicab group consists of the isometries of the plane with respect to the Taxicab metric defined by $d(u, v) = |u_x - v_x| + |u_y - v_y|$ where $u = (u_x, u_y)$, $v = (v_x, v_y)$ are vectors in $\mathbb{R}^2$ [2].

It has been shown in [2] that the group of isometries of the plane with respect to the taxicab metric is the semi-direct product of the dihedral group $D_4$ and $T(2)$ where $T(2)$ is the translation group. It has also been shown in [3] that $\text{Aut}(D_4) \cong A_f f (\mathbb{Z}_n) = ax + b$ | $a \in U, b \in \mathbb{Z}_n$

3 Proven Already

The cardinality of $\mathcal{H}$ is the cardinality of any hamel basis is the cardinality of $\mathbb{R}$

Definition 1. The Cauchy functional equation is a function $f : X \rightarrow Y$ such that $f(x + y) = f(x) + f(y) \forall x, y \in X$

Theorem 1. If an additive function $f$ satisfies the Cauchy functional equation and satisfies one of the following conditions, then there exists a real constant $c$ such that $f(x) = cx$ for all $x \in \mathbb{R}$ [1]:

1. $f$ is continuous at a point;
2. $f$ is monotonic on an interval of positive length;
3. $f$ is bounded from above or below on an interval of positive length;
4. $f$ is integrable;
5. $f$ is Lebesgue measurable;
6. $f$ is a Borel function.
4 What We Think:

\[ \text{Aut}(\mathbb{R}, +) \simeq \text{Aut}(\mathcal{H}, +) \] where \( \mathcal{H} \) is the collection of all Hamel bases.

5 Automorphism of \( \mathbb{R} \)

Definition 2. Definition: A set \( S \) finitely spans a vector space \( X \) if \( \forall x \in X \exists N \in \mathbb{N} \) such that

\[ x = \sum_{i=1}^{N} \alpha_i s_i \] where \( \alpha_i \) are scalars and \( s_i \in S \) \( \forall i \) (5.1)

Definition 3. A Hamel basis for an infinite vector space \( X \) is a linearly independent set that finitely spans \( X \).

Definition 4. An indexed Hamel basis is a list whose elements come from a Hamel basis such that the union of all elements in the indexed basis is the Hamel basis.

Theorem: Let \( H \) be a Hamel basis. If \( h \in H \), then \( h \in \mathbb{R}/\mathbb{Q} \)

Definition 5. Let \( H \) and \( H' \) be two indexed Hamel bases. Now let \( T_{H'} : \mathbb{R} \to \mathbb{R} \) be given by

\[ T_{H'} (x) = \sum_{i=1}^{N_x} \alpha_i T_{H'} (h_i) \] where \( T_{H'} (H) = H' \) (5.2)

We will call \( T \) a Hamel Basis Transformation. We will call \( H \) the Central basis for \( T_{H'} \). Now, denote the set of all Hamel Basis Transformations having a given Central Basis \( H \) by \( T_{H} \).

Lemma 1. If \( T \) is a Hamel Basis Transformation, then \( T \) is a Linear Operator.

Proof: Let \( x, y \in \mathbb{R} \) and \( p_x, p_y \in \mathbb{Q} \)

\[ T (p_x x + p_y y) = T \left( \sum_{i=0}^{N_x} q_{x,i} h_i + \sum_{i=0}^{N_y} q_{y,i} h_i \right) \] (5.3)

Without loss of generality, we can assume \( N_x = N_y = N \). If not, for \( M = \min\{N_x, N_y\} \) set \( q_{m,i} = 0 \forall i \in (M, \max\{N_x, N_y\}) \). Now,

\[ T (p_x x + p_y y) = \sum_{i=0}^{N} (p_x q_{x,i} + p_y q_{y,i}) T (h_i) \]

\[ = p_x T (x) + p_y T (y) \]
This shows that $T$ is a linear operator.

**Lemma 2.** If $T$ is a Hamel Basis Transformation, then $T \in \text{Aut } (\mathbb{R}, +)$

**Proof.** Let $H$ be the Central Basis for a Hamel Basis Transformation $T_{H'}$. Take $T_{H'} \in T_{3c}$ and for simplicity call it $T$.

Assume $r \neq 0$, then $r = \sum_{i=0}^{N_r} q_i h_i$ where $q_i \neq 0 \forall i$.

$$T(r) = \sum_{i=0}^{N_r} q_i T(h_i) = \sum_{i=0}^{N_r} q_i h'_i \text{ where } h'_i \in H'$$

(5.4)

and $q_i h'_i \neq 0$ since $0 \notin H'$. But if $\sum_{i=0}^{N_r} q_i h'_i = 0$, then $h'_i$ is not linearly independent since $q_i \neq 0 \forall i$, therefore $\{h'_i\}_{i=0}^{N_r}$ is not subset of any basis. Hence, $H'$ is not a basis which is a contradiction. Thus, $T(r) \neq 0$. By 2.2-10, [4], and since $T$ is a linear operator, the inverse map exists. Therefore $T$ is bijective.

**Lemma 3.** Let $T_{3c}$ have $H$ as a Central Basis. Now, if $\phi \in \text{Aut } (\mathbb{R}, +)$, then $\phi \in T_{3c}$

**Proof.** Now, we show that if $\phi \in \text{Aut } (\mathbb{R}, +)$ then $\phi \in T_{3c}$

By work of Cauchy and others[1], it is well known that for any $q_1 \in \mathbb{Q}$ and any $r_1 \in \mathbb{R}$ we have

$$\phi(qr_1) = q\phi(r_1)$$

(5.5)

furthermore, if $r_1$ is not a rational multiple of $r_2 \in \mathbb{R}$ then $\forall q_2 \in \mathbb{Q}$

$$\phi(q_1 r_1 + q_2 r_2) = q_1 \phi(r_1) + q_2 \phi(r_2)$$

(5.6)

We must show that $\phi(r_1) \neq q \phi(r_2)$ for some $q \in \mathbb{Q}$. Assume $\phi(r_1) = \phi(qr_2)$, then $\phi(r_1) - \phi(qr_2) = \phi(r_1 - qr_2) = 0$ But since $\phi$ is an automorphism, then $r_1 - qr_2 = 0$ which implies $r_1 = qr_2$ which is a contradiction to the assumption that $r_1$ is not a rational multiple of $r_2$.

By induction, if $S$ is a linearly independent subset of $\mathbb{R}$ and $r_i \in S \forall i$ and $q_i \in \mathbb{Q} \forall i$

$$\phi \left( \sum_{i=1}^{N} q_i r_i \right) = \sum_{i=1}^{N} q_i \phi (r_i)$$

(5.7)

We must show that $\phi(S)$ is a linearly independent set. Assume $\phi(S)$ is not linearly independent, then for some $r \in S$, $\phi(r) = \sum_{i=1}^{M} q_i \phi(r_i)$ where $\forall i q_i \neq 0$ and $r \neq r_i$. Then by above equation, $\phi \left( r - \sum_{i=1}^{M} q_i r_i \right) = 0$. This implies that
\[ r = \sum_{i=1}^{M} q_i r_i \] since \( \phi \) is an automorphism. Which is a contradiction to the assumption that \( S \) is a linearly independent subset of \( \mathbb{R} \).

By Zorn’s lemma, there exists a maximal such \( S \) such that \( S \) finitely spans \( \mathbb{R} \). Therefore, \( S \) is a Hamel basis for \( \mathbb{R} \).

By the properties of \( \phi \), it is a linear operator and therefore is uniquely determined by the images of the basis elements. Let \( H \) be a indexed hamel basis. Assume \( H' = \phi(H) \) does not span \( \mathbb{R} \), then \( \exists r' \in \mathbb{R} \) such that \( r' \) is not a linear combination of \( H' \) but \( \phi \) is completely determined by \( \phi(H) \) which implies there is no \( r \) in \( \mathbb{R} \) such that \( r' = \phi(r) \) and therefore \( \phi \) is not onto. This is a contradiction and implies that \( \phi(H) \) spans \( \mathbb{R} \).

Now, since \( \phi(H) \) is linearly independent and spans \( \mathbb{R} \), it forms a hamel basis for \( \mathbb{R} \).

\[ \Box \]

---

**Theorem 2.** From a indexed hamel basis \( H \) to \( H' \), there exists a unique indexed hamel basis Transformation.

Theorem: The group of all permutations of any indexed Hamel basis is a subgroup of \( \text{Aut}(\mathbb{R},+) \) is isomorphic to \( S(H) \).

---

**6 Matrix Representation**

**Definition 6.** The matrix index set \( I \) for a matrix \( M = [m_{ij}] \) is the set from which \( i \) and \( j \) come.

**Definition 7.** Let \( G \) be a matrix group and \( S \) be any set subset of the matrix index set for \( G \). Now, the filter of \( G \) over \( S \) is the set.

\[ G_S = \{ M = [m_{ij}] \in G \mid m_{ij} = 0 \ \forall \ i \text{ or } j \notin S \text{ except when } i = j \} \quad (6.1) \]

**Theorem 3.** The filter \( H \) of a matrix group \( G \) over \( S \) is a subgroup of \( G \).

Proof. Let \( A = [a_{ij}], B = [b_{ij}], \) and \( C = [c_{ij}] = AB \). Let \( I \) be the index set for \( G \). Now, the identity is in \( H \) and \( H \) is associative. The proof is left to the reader. We must now show that \( H \) is closed. If \( i = j \) or \( i, j \in S \), then \( c_{ij} \) can be anything. Assume \( i \neq j \) and \( i \notin S \), then \( c_{ij} = \sum_{k \in I} a_{ik}b_{kj} = a_{ii}b_{ij} \) since \( a_{ik} = 0 \ \forall \ k \neq i \). Since \( i \neq j \), then \( c_{ij} = 0 \). Similarly, if \( i \neq j \) and \( j \notin S \), then \( c_{ij} = 0 \).

Now we show \( A^{-1} = [\alpha_{ij}] \) inverse of \( A \in H \). It is known that for any matrix \( M \), \( M^{-1} = \frac{1}{\det(M)} F^T \) where \( F_{ij} = [f_{ij}] \) is the \((i, j)\)th minor of \( A \). Now, as before if \( i = j \) or \( i, j \in S \), then \( \alpha_{ij} \) can be anything. Now, assume \( i \neq j \) and \( i \notin S \), then \( \alpha_{ij} = 0 \). \( \Box \)
Theorem 4. If there exists a maximal $H$ subgroup of $\text{Aut}(\mathbb{R})$, then $H$ is uncountable.

-------------------

Theorem: The $\text{Inn}(\mathbb{R})$ is the identity map.

Theorem: The $\text{Out}(\mathbb{R}) = \text{Aut}(\mathbb{R})$

Theorem: The center of $\mathbb{R}$ is the set of all linear maps.

-------------------

TO DO: 1.) identify subgroups of $\text{Aut}(\mathbb{R})$ - are they maximal? - are they normal? if so $\text{Aut}(\mathbb{R})/N$ is a group 2.) identify elements of order 2 in $\text{Aut}(\mathbb{R})$ - centralizer of these elements: $C_a(G) = \{g \in G | ga = ag\}$ which is the Fixed group of $a$ in $G$. 

5
References

   Published in USA by publication service.

