$k$-involutions of algebraic groups of type $G_2$

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A $k$-involution, $\theta : G \to G$, is an automorphism of order 2 that is defined over $k$.

Let $G$ be a reductive connected algebraic group defined over $k$ and let $\theta \in \text{Aut}(G)$ be a $k$-involution. Then a symmetric $k$-variety will be the quotient

$$G_k/H_k \cong Q_k = \{x\theta(x)^{-1} \mid x \in G_k\}.$$
symmetric $k$-varieties

A torus $S$ is $\theta$-split if $\theta(s) = s^{-1}$ for all $s \in S$.

A torus is $(\theta, k)$-split if it is both $\theta$-split and $k$-split.

$I_s(x) = sx{s^{-1}}$

$X^*(T)$ is the group of characters associated to a torus $T$

$\Phi(T)$ is the root space associated to a torus $T$
A.G. Helminck simplifies the classification of $k$-involutions of $k$-split algebraic groups into the classification of the following invariants,

(1) classification of admissible $k$-involutions of

$$(X^*(T), X^*(S), \Phi(T), \Phi(S)),$$

letting $T$ be a maximal torus, $T \supset S$ is a maximal $k$-split torus, $\Phi$ denotes the roots associated with a torus,

(2) classification of the $G_k$-isomorphy classes of $k$-inner elements

$s \in I_k(S_{\theta}^-)$, where the set $S_{\theta}^- = \{ s \in S \mid \theta(s) = s^{-1} \}$.

All $k$-involutions are of the form $\theta \circ I_s$ where $\theta$ is an invariant of type (1) and $s$ is of type (2).
Theorem

The automorphism group $\text{Aut}(C_K)$ of the octonion algebra $C_K$ is a connected, simple algebraic group of type $G_2$.

Proposition

Let $C$ be an octonion algebra over $k$. Then the automorphism group $\text{Aut}(C)$ is defined over $k$.

from now on we think of our algebraic groups of type $G_2$ as automorphism groups of octonion algebras
composition algebras

A composition algebra, $C$, over a field, $k$, is a potentially nonassociative algebra with identity element, $e$, such that there exists a nondegenerate quadratic form $N$ on $C$ where

$$N(xy) = N(x)N(y), \quad x, y \in C$$

$$\langle x, y \rangle = N(x + y) - N(x) - N(y),$$

with algebra involution

$$\overline{x} = \langle x, e \rangle e - x.$$
If $D$ is a split quaternion algebra, $D \cong M_2(k)$, the $2 \times 2$ matrices over $k$ with typical matrix multiplication, and determinant as its quadratic form.

$$
\begin{bmatrix}
\alpha_{11} & \alpha_{12} \\
\alpha_{21} & \alpha_{22}
\end{bmatrix}
= 
\begin{bmatrix}
\alpha_{22} & -\alpha_{12} \\
-\alpha_{21} & \alpha_{11}
\end{bmatrix}
$$
An element of our split octonion algebra will be an ordered pair of elements from a quaternion algebra, \((x, y)\) where \(x, y \in D\).

\[
N((x, y)) = \det(x) - \det(y)
\]

\[
(x, y)(u, v) = (xu + \overline{vy}, vx + y\overline{u})
\]
When $\text{Aut}(C)$ is split there is a $k$-split maximal torus $T \subset \text{Aut}(C)$ of the form

$$T = \{ t_{(\gamma,\delta)} \equiv \text{diag}(1, \gamma \delta, (\gamma \delta)^{-1}, 1, \delta^{-1}, \gamma, \gamma^{-1}, \delta) \mid \gamma, \delta \in k^* \}.$$ 

**Proposition**

The map $\mathcal{I}_{g^*}$ is an automorphism of order 2 of $\text{Aut}(C)$ where $g^* = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \in \text{Aut}(C)$, and $\mathcal{I}_{g^*}(t) = t^{-1}$ for all $t \in T$. 
Proposition [Jacobson, 58]

For $s, t \in \text{Aut}(C)$ such that $s^2 = t^2 = \text{id}$, $t \cong s$ if and only if $s$ and $t$ fix elementwise isomorphic quaternion subalgebras.

The algebraic groups we are considering have trivial centers, and no non-trivial outer automorphisms.

Corollary

For $s, t \in \text{Aut}(C)$ $I_s^2 = I_t^2 = \text{id}$ then $I_t \cong I_s$ if and only if $s$ and $t$ leave isomorphic quaternion subalgebras invariant.
Pfister forms

Proposition

The structure of a composition algebra is determined by its norm.

Its norm is determined by its Pfister form.

Example:

\[
\left( \frac{\lambda_1, \lambda_2}{k} \right) \sim N_D
\]

where \(e, a, b, ab\) are a basis for \(D\) a quaternion algebra and \(a^2 = \lambda_1\) and \(b^2 = \lambda_2\).
Example: Let $g_* \in \text{Aut}(C)$ fixes elementwise the following quaternion algebra

$$k \left[ \begin{array}{c} 1 \\ 1 \\ e \end{array} , 0 \right] \bigoplus k \left[ \begin{array}{c} 1 \\ 1 \\ a \end{array} , 0 \right] \bigoplus k \left( 0, \begin{array}{c} 1 \\ 1 \\ b \end{array} \right) \bigoplus k \left( 0, \begin{array}{c} 1 \\ 1 \\ ab \end{array} \right),$$

and $(b + ab)(e + a + b + ab) = 0$, which tells us the quaternion subalgebra is split.

So $\mathcal{I}_{g_*}$ is a representative of the conjugacy class in $\text{Aut}(C)$ that fixes a split quaternion subalgebra.

Over any field $k$ there is only one such conjugacy class of quaternion algebras, and so only one class of $k$-involutions.
Theorem

- When \( k = K \) and \( \mathbb{F}_p \) with \( p > 2 \), there is only one isomorphism class of \( k \)-involutions of \( \text{Aut}(C) \).
- When \( k = \mathbb{R} \) and \( \mathbb{Q}_2 \), \( \theta \) and \( \theta \circ \mathcal{I}_{t(1,-1)} \) are representatives of the two isomorphism classes of \( k \)-involutions of \( G = \text{Aut}(C) \).
- When \( k = \mathbb{Q}_p \) with \( p \neq 2 \), \( \theta \) and \( \theta \circ \mathcal{I}_{t(-N_p,-pN_p^{-1})} \) give us the two isomorphism classes.
- When \( k = \mathbb{Q} \) we have an infinite number of non-isomorphic quaternion division subalgebras, and so an infinite number of classes of \( k \)-involutions.
fixed point groups

Proposition

Let $t \in \text{Aut}(C) = G$ such that $t^2 = \text{id}$ and $D \subset C$ the quaternion algebra elementwise fixed by $t$ then $f \in G^I_t = \{ g \in G \mid I_t(g) = g \}$ if and only if $f$ leaves $D$ invariant.

In our case this always is the subgroup containing elements of the form

$$t((x, y)) = (cx^c, pcy^c),$$

where $c, x, y, p \in D$ and $N_D(p) = 1$. 
fixed point groups

Example: for $k = \mathbb{R}$ or $\mathbb{Q}_2$ we have two isomorphism classes

$g* \in \operatorname{Aut}(C)$ fixes elementwise a split quaternion subalgebra, so the class $[I_{g*}]$ has the fixed point group $\operatorname{PGL}_2(k) \times \operatorname{SL}_2(k)$.

$g*t_{(1,-1)} \in \operatorname{Aut}(C)$ leaves a quaternion division algebra fixed elementwise, so the class $[I_{g*t_{(1,-1)}}] = [I_{g*} \circ I_{t_{(1,-1)}}]$ has the fixed point group $\operatorname{SO}(D_0, N_D) \times \operatorname{Sp}(1)$. 
Galois cohomology

According to Serre...

\[ H^0(\text{Gal}_k, \text{Aut}(X, K)) = \text{Aut}(X, k) \]

If we consider \( H^1(\text{Gal}_k, \text{Aut}(X, K)) \) we are considering the \( K/k \)-forms of \( X \), where a \( K/k \)-form of \( X \) will be any object \( Y \) defined over \( k \) that becomes isomorphic to \( X \) when the ground field of \( X \) and \( Y \) are extended to \( K \).
Galois cohomology

The elements of $H^1(Gal_k, \text{Aut}(C, D, K))$ correspond to the isomorphism classes of $k$-involutions.

Let $G = \text{Aut}(C)$, $C_k$ be the the isomorphism classes of $k$-involutions and $Z_G(C_k)$ be the set of isomorphism classes of centralizers of elements of order 2 in Aut($G$).

There is a bijection between $H^1(Gal_k, \text{Aut}(C, D, K))$ and $H^1(Gal_k, Z_G(C_K))$.

Proposition

The following map is bijective

$$Z_G : C_k \rightarrow H^1(Gal_k, Z_G(C_K)),$$

where $[\theta] \mapsto [Z_G(\theta)]$. 